

Universality of quantum symplectic structure

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Abstract

Operating in the framework of ‘supmech’ (a scheme of mechanics which aims at providing a concrete setting for the axiomatization of physics and of probability theory as required in Hilbert’s sixth problem; integrating noncommutative symplectic geometry and noncommutative probability in an algebraic setting, it associates, with every ‘experimentally accessible’ system, a symplectic algebra and operates essentially as noncommutative Hamiltonian mechanics with some extra sophistication in the treatment of states) it is shown that interaction between systems can be consistently described only if either (i) all system algebras are commutative or (ii) all system algebras are noncommutative and have a quantum symplectic structure characterized by a *universal* Planck type real-valued constant of the dimension of action.

*Like it or not
If you are noncommutative
You have no option
But to be quantum.*

Introduction

Two (closely related) great intellectual challenges before theoretical physicists are :

- (i) Construction of the most economical and complete description of nature (theory of ‘everything’);
- (ii) Solution of Hilbert’s sixth problem [23] (axiomatization of physics and probability theory).

For solving both these problems, two possible strategies are :

- (a) Solve (i), then brush up the formalism and axiomatize so as to solve (ii).
- (b) Solve (i) in such a manner that (ii) is automatically solved (essentially integrating the two problems).

The author’s preference is for (b), mainly because, in this case, relatively clearer thinking about (and contact with) fundamentals is expected to prevail.

The adoption of (b) instead of (a) (which reflects the prevalent attitude) implies a change in outlook and priorities. It puts greater emphasis on the development of an ‘appropriate’ formalism. Without entering into a detailed discussion about the term ‘appropriate’, we shall take it to mean that the formalism should be reasonably broad-based so as to cover *all* systems in nature, it should employ mathematics best suited for the development of the adopted ideas and concepts and should be self consistent.

In the present era in physics, quantum theory is believed to be applicable to all systems in nature. As far as experimental predictions are concerned, it has been eminently successful. It is, however, in need of a satisfactory formalism which should be in the nature of its autonomous development (as opposed to the traditional practice of *quantizing* classical systems) and which should provide for a satisfactory treatment of measurements on quantum systems without introducing ad hoc assumptions like the von Neumann reduction postulate.

The desired ‘appropriate’ formalism must do justice to the basic features of quantum mechanics (QM) : the noncommutative kinematics of observables and its intrinsically probabilistic nature as reflected in the behavior of quantum states. The latter aspect, traditionally referred to as ‘quantum probability’ has been explored in several versions [30], [26], [33], [27], [25], [35], [1], [31], [28]. The one best suited to our needs is the one [28] based on

complex, associative, unital (i.e. having a unit element) and not necessarily commutative $*$ -algebras (henceforth referred to as ALGEBRAS). In this version, quantum probability may be referred to as noncommutative probability. (Not. Since the term ‘noncommutative measure theory’ has been used for the algebraic development based on von Neumann algebras presented in, for example, Connes’ book [7], one might take ‘noncommutative probability’ to mean its ‘normalized’ sub-domain; we shall, however, reserve this term for the more general algebraic version of Ref.[28].

The two (mutually related) noncommutative developments relating to observables and states may be jointly referred to as the ‘noncommutative culture’ of QM.

Heisenberg’s [22] idea –that kinematics underlying QM must be based on a noncommutative algebra of observables - was incorporated into a scheme of mechanics (called ‘matrix mechanics’) by Born, Jordan, Dirac and Heisenberg [4], [12], [5]. The proper geometrical framework for the construction of the ‘quantum Poisson brackets’ of this mechanics is provided by noncommutative symplectic geometry based on the derivation -based differential calculus developed by Dubois-Violette and coworkers [16], [19], [17], [18], [14]; the latter will be referred to as DVNCG.

Both, the noncommutative probability and DVNCG employ ALGEBRAS which are, therefore, the natural domain for the development of the ‘non-commutative culture’ mentioned above. It makes perfect sense to develop a coherent scheme of mechanics integrating noncommutative symplectic geometry and noncommutative probability in the setting of ALGEBRAS. Such a mechanics (called ‘supmech’) has been developed by the author. It has QM and classical Hamiltonian mechanics as special subdisciplines and is projected as the appropriate framework for an autonomous development of QM. The detailed development of this mechanics will be presented elsewhere [11]. Here we shall restrict ourselves to a reasonably self-contained presentation of a development (within the domain of supmech) of some special theoretical interest : a consistent description of interaction between systems in the supmech framework is possible only if either

- (i) all the system ALGEBRAS are commutative, or
- (ii) all system ALGEBRAS are non-commutative and have a quantum symplectic structure characterized by a *universal* real-valued constant of the dimension of action.

The formalism, therefore, has a natural place for the Planck constant as a universal constant — just as special relativity has a natural place for a universal speed. In fact, the situation in supmech is somewhat better because, whereas in special relativity, the existence of a universal speed is *postulated*, in supmech the existence of a universal Planck like constant is *dictated/predicted* by the formalism.

Plan of the paper. In section 1, a brief account of DVNCG is given which includes a discussion of its generalization [8] involving algebraic pairs $(\mathcal{A}, \mathcal{X})$ where \mathcal{A} is an ALGEBRA and \mathcal{X} a Lie subalgebra of $\text{Der}(\mathcal{A})$ and of the mappings [8], [9] induced on derivations by the *-algebra isomorphisms (analogues of the push-forward and pull-back mappings induced by diffeomorphisms on vector fields and differential forms. In section 2, the ‘noncommutative culture’ of Hilbert space QM is expressed in algebraic terms [to conform to the noncommutative geometry (NCG)- based developments of the next section]. In section 3, an outline of the supmech formalism is presented adequate for the treatment of interacting systems in supmech in the next section. The last section contains some concluding remarks.

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1. Derivation based differential calculus

1.1 *Noncommutative differential forms.* The central object in DVNCG is an ALGEBRA \mathcal{A} ; its elements will be denoted as A, B, \dots and the identity element as I . The *-operation (or involution) $*$: $\mathcal{A} \rightarrow \mathcal{A}$ is an antilinear mapping satisfying the relations

$$(AB)^* = B^* A^*, \quad (A^*)^* = A, \quad I^* = I.$$

An element $A \in \mathcal{A}$ is called hermitian if $A^* = A$. The center $Z(\mathcal{A})$ of \mathcal{A} is the set of those elements of \mathcal{A} which commute with all elements of \mathcal{A} .

A derivation of \mathcal{A} is a linear map $X : \mathcal{A} \rightarrow \mathcal{A}$ such that $X(AB) = X(A)B + AX(B)$. Introducing the multiplication operator μ on \mathcal{A} defined as $\mu(A)B = AB$, the condition that X is a derivation may be expressed as

$$(1) \quad X \circ \mu(A) - \mu(A) \circ X = \mu(X(A)).$$

The set $Der(\mathcal{A})$ of all derivations of \mathcal{A} is a Lie algebra with the Lie bracket $[X, Y] = X \circ Y - Y \circ X$. The inner derivations D_A defined by $D_A B = [A, B]$ satisfy the relation

$$[D_A, D_B] = D_{[A, B]}$$

and constitute a Lie subalgebra $IDer(\mathcal{A})$ of $Der(\mathcal{A})$.

In DVNCG it is implicitly assumed that the ALGEBRAS being employed have a reasonably rich supply of derivations so that various constructions involving them have a nontrivial content.

An involution $*$ on $Der(\mathcal{A})$ is defined by the relation $X^*(A) = [X(A^*)]^*$. We have the (easily verifiable) relations

$$[X, Y]^* = [X^*, Y^*], \quad (D_A)^* = -D_{A^*}.$$

By a *differential calculus* on \mathcal{A} one means a formalism involving differential form like objects on \mathcal{A} with analogues of exterior product, exterior derivative and involution defined on them. For noncommutative \mathcal{A} , it is not unique; a systematic discussion of the variety of choices may be found in Ref.[17]. In applications of NCG one makes a choice according to convenience. In DVNCG (which is best suited for a geometrical treatment of QM) one employs a derivation-based differential calculus in which the spaces of differential p-forms are (a subclass—to be specified later—of) Chevalley-Eilenberg p-cochain spaces $C^p(Der(\mathcal{A}), \mathcal{A})$ [36]. Such a p-cochain ω is, for $p \geq 1$, a multilinear map of $[Der(\mathcal{A})]^p$ into \mathcal{A} which is skew-symmetric :

$$\omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}) = \kappa_\sigma \omega(X_1, \dots, X_p)$$

where κ_σ is the parity of the permutation σ ; we have $C^0(Der(\mathcal{A}), \mathcal{A}) = \mathcal{A}$.

An involution $*$ on the cochains is defined by the relation $\omega^*(X_1, \dots, X_p) = [\omega(X_1^*, \dots, X_p^*)]^*$; ω is said to be real (imaginary) if $\omega^* = \omega$ ($\omega^* = -\omega$).

The exterior product

$$\wedge : C^p(Der(\mathcal{A}), \mathcal{A}) \times C^q(Der(\mathcal{A}), \mathcal{A}) \rightarrow C^{p+q}(Der(\mathcal{A}), \mathcal{A})$$

is defined as in the commutative case :

$$(2) \quad \begin{aligned} (\alpha \wedge \beta)(X_1, \dots, X_{p+q}) &= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \kappa_\sigma \alpha(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \cdot \\ &\quad \cdot \beta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}). \end{aligned}$$

With this product, the N_0 -graded vector space (where N_0 is the set of non-negative integers)

$$C(Der(\mathcal{A}, \mathcal{A})) = \bigoplus_{p \geq 0} C^p(Der(\mathcal{A}, \mathcal{A}))$$

becomes a graded complex algebra.

The Lie algebra $Der(\mathcal{A})$ acts on itself and on $C(Der(\mathcal{A}, \mathcal{A}))$ through *Lie derivatives*. For each $Y \in Der(\mathcal{A})$, one defines linear mappings $L_Y : Der(\mathcal{A}) \rightarrow Der(\mathcal{A})$ and $L_Y : C^p(Der(\mathcal{A}), \mathcal{A}) \rightarrow C^p(Der(\mathcal{A}), \mathcal{A})$ such that the following three conditions hold :

$$(3) \quad L_Y(A) = Y(A) \quad \text{for all } A \in \mathcal{A}$$

$$(4) \quad L_Y[X(A)] = (L_Y X)(A) + X[L_Y(A)]$$

$$(5) \quad \begin{aligned} L_Y[\omega(X_1, \dots, X_p)] &= (L_Y \omega)(X_1, \dots, X_p) \\ &+ \sum_{i=1}^p \omega(X_1, \dots, X_{i-1}, L_Y X_i, \dots, X_p). \end{aligned}$$

The first two conditions give

$$(6) \quad L_Y(X) = [Y, X]$$

which, along with the third, gives

$$(7) \quad \begin{aligned} (L_Y \omega)(X_1, \dots, X_p) &= Y[\omega(X_1, \dots, X_p)] \\ &- \sum_{i=1}^p \omega(X_1, \dots, X_{i-1}, [Y, X_i], \dots, X_p). \end{aligned}$$

Some important properties of the Lie derivative are, in obvious notation,

$$(8) \quad [L_X, L_Y] = L_{[X, Y]}$$

$$(9) \quad L_Y(\alpha \wedge \beta) = (L_Y \alpha) \wedge \beta + \alpha \wedge (L_Y \beta).$$

For any $X \in Der(\mathcal{A})$, we define the interior product $i_X : C^p(Der(\mathcal{A}), \mathcal{A}) \rightarrow C^{p-1}(Der(\mathcal{A}), \mathcal{A})$ (for $p \geq 1$) by

$$(10) \quad (i_X \omega)(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1})$$

and $i_X(A) = 0$ for all $A \in \mathcal{A}$. The following relations involving the Lie derivative and the interior product hold (here α is a p-form)

$$(11) \quad i_X \circ i_Y + i_Y \circ i_X = 0$$

$$(12) \quad i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X \beta)$$

$$(13) \quad L_X \circ i_Y - i_Y \circ L_X = i_{[X,Y]}.$$

The exterior derivative $d : C^p(Der(\mathcal{A}), \mathcal{A}) \rightarrow C^{p+1}(Der(\mathcal{A}), \mathcal{A})$ is defined through the relation

$$(14) \quad (i_X \circ d + d \circ i_X)\omega = L_X \omega.$$

This equation determines the operation of d on cochains of various degrees recursively. For $p = 0$, it takes the form

$$(15) \quad (dA)(X) = X(A).$$

and, for general $p \geq 0$,

$$(16) \quad \begin{aligned} & (d\alpha)(X_0, X_1, \dots, X_p) \\ &= \sum_{i=0}^p (-1)^i X_i[\alpha(X_0, \dots, \hat{X}_i, \dots, X_p)] \\ &+ \sum_{0 \leq i < j \leq p} (-1)^j \alpha(X_0, \dots, X_{i-1}, [X_i, X_j], X_{i+1}, \dots, \hat{X}_j, \dots, X_p) \end{aligned}$$

where the hat indicates omission. The exterior derivative satisfies the nilpotency condition $d^2 = 0$ and the relations

$$(17) \quad d \circ L_Y = L_Y \circ d$$

$$(18) \quad d(\alpha \wedge \beta) = d\alpha + \alpha \wedge (d\beta).$$

The nilpotency of d implies that the pair $(C(Der(\mathcal{A}), \mathcal{A}), d)$ constitutes a cochain complex. We shall call a cochain α closed if $d\alpha = 0$ and exact if $\alpha = d\beta$ for some cochain β .

Following Ref.[17], we consider the subset $\Omega(\mathcal{A})$ of $C(Der(\mathcal{A}), \mathcal{A})$ consisting of $Z(\mathcal{A})$ -linear cochains which means the cochains α satisfying the condition

$$(19) \quad \alpha(\dots, KX, \dots) = K\alpha(\dots, X, \dots)$$

for all $X \in \text{Der}(\mathcal{A})$ and $K \in Z(\mathcal{A})$. This subset is closed under the d-operation as can be easily easily verified using the relation

$$(20) \quad [X, KY] = X(K)Y + K[X, Y]$$

for all $X, Y \in \text{Der}(\mathcal{A})$ and $K \in Z(\mathcal{A})$. We shall reserve the term ‘differential forms’ for elements of $\Omega(\mathcal{A})$. We have

$$\Omega(\mathcal{A}) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{A})$$

with $\Omega^0(\mathcal{A}) = \mathcal{A}$. Elements of $\Omega^p(\mathcal{A})$ will be called differential p-forms.

1.2 Induced mappings on derivations and differential forms

A *-algebra isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ induces a mapping $\Phi_* : \text{Der}(\mathcal{A}) \rightarrow \text{Der}(\mathcal{B})$ given by

$$(21) \quad (\Phi_* X)(B) = \Phi(X[\Phi^{-1}(B)])$$

for all $X \in \text{Der}(\mathcal{A})$ and $B \in \mathcal{B}$. It is the analogue (and a generalization) of the mapping induced by a diffeomorphism on vector fields and satisfies the expected relations (with $\Psi : \mathcal{B} \rightarrow \mathcal{C}$)

$$(22) \quad (\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*; \quad \Phi_*[X, Y] = [\Phi_* X, \Phi_* Y].$$

It is easily seen that Φ_* is a Lie-algebra isomorphism.

The *-isomorphism Φ also induces a mapping

$$\Phi^* : C^p(\text{Der}(\mathcal{B}), \mathcal{B}) \rightarrow C^p(\text{Der}(\mathcal{A}), \mathcal{A})$$

given, in obvious notation, by

$$(23) \quad (\Phi^* \omega)(X_1, \dots, X_p) = \Phi^{-1}[\omega(\Phi_* X_1, \dots, \Phi_* X_p)].$$

These mappings are analogues (and generalizations) of the pull-back mappings on traditional differential forms induced by diffeomorphisms. It is easily seen that the mapping Φ_* preserves $Z(\mathcal{A})$ -linear combinations of derivations and that Φ^* maps differential forms onto differential forms. The following expected relations hold :

$$(24) \quad (\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$$

$$(25) \quad \Phi^*(\alpha \wedge \beta) = (\Phi^* \alpha) \wedge (\Phi^* \beta)$$

$$(26) \quad \Phi^*(d\alpha) = d(\Phi^* \alpha).$$

Let $\Phi_t : \mathcal{A} \rightarrow \mathcal{A}$ be a one-parameter family of transformations (i.e. ALGEBRA-automorphisms) given, for small t , by

$$\Phi_t(A) \simeq A + tg(A)$$

where g is some linear mapping of \mathcal{A} into itself. The condition

$$\Phi_t(AB) = \Phi_t(A)\Phi_t(B)$$

gives $g(AB) = g(A)B + Ag(B)$ implying that $g(A) = Y(A)$ for some $Y \in \text{Der}(\mathcal{A})$; we call Y the infinitesimal generator of the one-parameter family Φ_t . It is easily verified that the infinitesimal transformations of derivations and of p-forms induced by Φ_t are given by the respective Lie derivatives :

$$(27) \quad (\Phi_t)_* X \simeq X + tL_Y X$$

$$(28) \quad (\Phi_t)^* \omega \simeq \omega - tL_Y \omega.$$

1.3 Symplectic structures

A *symplectic structure* on an ALGEBRA \mathcal{A} is defined as a differential 2-form ω (the *symplectic form*) which is (i) closed and (ii) non-degenerate in the sense that, for every $A \in \mathcal{A}$, there is a unique derivation Y_A in $\text{Der}(\mathcal{A})$ [the (globally) Hamiltonian derivation corresponding to A] such that

$$(29) \quad i_{Y_A} \omega = -dA.$$

The pair (\mathcal{A}, ω) is called a *symplectic algebra*.

A *symplectic mapping* from a symplectic algebra (\mathcal{A}, α) to another one (\mathcal{B}, β) is an ALGEBRA-isomorphism (i.e a *-algebra isomorphism mapping the unit element of \mathcal{A} to the unit element of \mathcal{B}) such that $\Phi^* \beta = \alpha$. A symplectic mapping from a symplectic algebra onto itself will be called a *canonical/symplectic transformation*. The symplectic form and its exterior powers are invariant under canonical transformations.

Given a symplectic algebra (\mathcal{A}, ω) , the *Poisson bracket* (PB) of two elements A and B of \mathcal{A} is defined as

$$(30) \quad \{A, B\} = \omega(Y_A, Y_B) = Y_A(B) = -Y_B(A).$$

It obeys the Leibnitz rule :

$$(31) \quad \begin{aligned} \{A, BC\} &= Y_A(BC) = Y_A(B)C + BY_A(C) \\ &= \{A, B\}C + B\{A, C\}. \end{aligned}$$

As in the classical case [41], we also have the other two properties of PBs :

(i) The Jacobi identity holds :

$$(32) \quad \begin{aligned} 0 &= \frac{1}{2}(d\omega)(Y_A, Y_B, Y_C) \\ &= \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\}; \end{aligned}$$

this makes (along with bilinearity and antisymmetry of the PBs) the pair $(\mathcal{A}, \{, \})$ a Lie algebra.

(ii) The correspondence $A \rightarrow Y_A$ is a Lie -algebra homomorphism from the above Lie algebra into $Der(\mathcal{A})$:

$$(33) \quad [Y_A, Y_B] = Y_{\{A, B\}}.$$

An element A of \mathcal{A} can act, via Y_A , as the infinitesimal generator of a one-parameter family of canonical transformations. The change in $B \in \mathcal{A}$ due to such an infinitesimal transformation is

$$(34) \quad \delta B = \epsilon Y_A(B) = \epsilon \{A, B\}.$$

1.4 Canonical symplectic structure on ‘special’ ALGEBRAS

An ALGEBRA will be called *special* if it has a trivial center and if all its derivations are inner. The differential 2-form ω_c defined on such an algebra \mathcal{A} by

$$(35) \quad \omega_c(D_A, D_B) = [A, B]$$

is said to be the *canonical form* of \mathcal{A} . (This differs from the definition in Ref.[16], [17] by a factor of i.) It is easily seen to be closed [the equation $(d\omega_c)(D_A, D_B, D_C) = 0$ is nothing but the Jacobi identity for the commutator], imaginary (i.e. $\omega_c^* = -\omega_c$) and dimensionless. For any $A \in \mathcal{A}$, the equation

$$\omega_c(Y_A, D_B) = -(dA)(D_B) = [A, B]$$

(for all $B \in \mathcal{A}$) has the unique solution $Y_A = D_A$; this gives

$$(36) \quad i_{D_A}\omega_c = -dA.$$

The form ω_c defines, on \mathcal{A} , the *canonical symplectic structure*; the corresponding PB is a commutator :

$$(37) \quad \{A, B\} = D_A(B) = [A, B].$$

Using Equations (36) and (14), it is easily seen that the form ω_c is *invariant* in the sense that $L_X \omega_c = 0$ for all $X \in \text{Der}(\mathcal{A})$. The invariant symplectic structure on the algebra $M_n(C)$ of complex $n \times n$ matrices obtained in Ref. [19] is a special case of the canonical symplectic structure on special ALGEBRAS described above.

If, on a special ALGEBRA \mathcal{A} , instead of ω_c , we take $\omega = b\omega_c$ as the symplectic form (where b is a nonzero complex number), we have

$$(38) \quad Y_A = b^{-1}D_A, \quad \{A, B\} = b^{-1}[A, B].$$

We shall see below that the so-called ‘quantum symplectic structure’ is such a symplectic structure with $b = -i\hbar$. Note that b must be imaginary to make ω real. Just to have a convenient name, we shall refer to the symplectic structure of the above sort (for general non-zero b) as a quantum symplectic structure with parameter b .

1.5 A generalization of the derivation-based differential calculus

A useful generalization of the formalism presented in this section so far is obtained by restricting the derivations to a Lie subalgebra \mathcal{X} of $\text{Der}(\mathcal{A})$; the central object in the whole development will now be, instead of the ALGEBRA \mathcal{A} , the pair $(\mathcal{A}, \mathcal{X})$. To get a feel for the implications of working with such a pair, we consider a couple of examples, one ‘commutative’ and the other ‘noncommutative’.

(i) $\mathcal{A} = C^\infty(R^3)$; \mathcal{X} = the Lie subalgebra of the Lie algebra $\mathcal{X}(R^3)$ of smooth vector fields on R^3 generated by the Lie differential operators $L_j = \epsilon_{jkl}x_k\partial_l$ for the $\text{SO}(3)$ -action on R^3 . These differential operators act on the 2-dimensional spheres that constitute the leaves of the foliation $R^3 - \{(0, 0, 0)\} \cong S^2 \times R$. Employing the polar coordinates (r, θ, ϕ) on R^3 (which are obviously adapted to the above-mentioned foliation), the variable r in the functions $f(r, \theta, \phi)$ in $C^\infty(R^3)$ will remain unaffected by the derivations in \mathcal{X} . It follows that the restriction to the pair $(\mathcal{A}, \mathcal{X})$ in the present case amounts to working on a leaf (S^2) of the above-mentioned foliation.

(ii) $\mathcal{A} = M_4(C)$, the algebra of complex 4×4 matrices. The vector space C^4 on which these matrices act serves as the carrier space of the spin $s = 3/2$ projective irreducible representation of the rotation group $SO(3)$. Denoting by $S_j (j = 1, 2, 3)$ the representatives of the generators of the Lie algebra $so(3)$ in this representation, let \mathcal{X} be the real Lie algebra generated by the inner derivations $D_{S_j} (j = 1, 2, 3)$. In the treatment of spin dynamics of a spin $s = 3/2$ object, one will effectively be using the pair $(\mathcal{A}, \mathcal{X})$.

In the generalized derivation-based differential calculus based on a pair $(\mathcal{A}, \mathcal{X})$, one has the derivations restricted to \mathcal{X} and the p-cochains are those in the space $C^p(\mathcal{X}, \mathcal{A})$; the corresponding differential p-form space will be denoted as $\Omega^p(\mathcal{X}, \mathcal{A})$. Obviously $\Omega^p(Der(\mathcal{A}), \mathcal{A}) \equiv \Omega^p(\mathcal{A})$.

To define the induced mappings Φ_* and Φ^* in the present context, one should employ a ‘pair isomorphism’ $\Phi : (\mathcal{A}, \mathcal{X}) \rightarrow (\mathcal{B}, \mathcal{Y})$ which consists of an ALGEBRA-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that the induced Lie algebra isomorphism $\Phi_* : Der(\mathcal{A}) \rightarrow Der(\mathcal{B})$ restricts to an isomorphism of \mathcal{X} onto \mathcal{Y} . Various properties of the induced mappings hold as before.

Given a one-parameter family of transformations (i.e. pair-automorphisms) $\Phi_t : (\mathcal{A}, \mathcal{X}) \rightarrow (\mathcal{A}, \mathcal{X})$, the condition $(\Phi_t)_* \mathcal{X} \subset \mathcal{X}$ implies that the infinitesimal generator Y of Φ_t must satisfy the condition $[Y, X] \in \mathcal{X}$ for all $X \in \mathcal{X}$. In practical applications, one will generally have $Y \in \mathcal{X}$ which obviously satisfies the above-mentioned condition.

The concept of a symplectic algebra (\mathcal{A}, ω) is now generalized to that of a ‘generalized symplectic algebra’ $(\mathcal{A}, \mathcal{X}, \omega)$ where now $\omega \in \Omega^2(\mathcal{X}, \mathcal{A})$. The non-degeneracy condition on ω now demands, for a given $A \in \mathcal{A}$, the existence of a unique derivation $Y_A \in \mathcal{X}$ such that Eq.(29) holds. A symplectic mapping $\Phi : (\mathcal{A}, \mathcal{X}, \alpha) \rightarrow (\mathcal{B}, \mathcal{Y}, \beta)$ is now an ALGEBRA-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that the induced mapping Φ_* restricts to an isomorphism of \mathcal{X} onto \mathcal{Y} and $\Phi^* \beta = \alpha$.

2. The noncommutative culture of quantum mechanics; the quantum symplectic structure

In this section, we shall present the traditional formalism of QM in a not-so-familiar algebraic setting so as to obtain a useful characterization of its ‘noncommutative culture’.

We start by considering the QM of a non-relativistic spinless particle. The central object in it is the Hilbert space $\mathcal{H} = L^2(R^3, dx)$ of complex square-

integrable functions on R^3 . The fundamental observables of such a particle are the Cartesian components $X_j, P_j (j = 1, 2, 3)$ of position and momentum vectors which are self-adjoint linear operators represented, in the oft-used Schrödinger representation, as

$$(39) \quad (X_j \phi)(x) = x_j \phi(x); \quad (P_j \phi)(x) = -i\hbar \frac{\partial \phi}{\partial x_j}.$$

These operators satisfy the canonical commutation relations (CCR)

$$(40) \quad [X_j, X_k] = 0 = [P_j, P_k]; \quad [X_j, P_k] = i\hbar I \quad (j, k = 1, 2, 3)$$

where I is the unit operator. The functions ϕ in Eq.(39) must be restricted to a suitable dense domain \mathcal{D} in \mathcal{H} which is generally taken to be the space $\mathcal{S}(R^3)$ of Schwartz functions. Other operators appearing in QM of the particle belong to the algebra \mathcal{A} generated by the operators $X_j, P_j (j = 1, 2, 3)$ and I [subject to the CCR (40)]. The space $\mathcal{D} = \mathcal{S}(R^3)$ is clearly an invariant domain for all elements of \mathcal{A} . Defining a *-operation on \mathcal{D} by $A^* = A^\dagger|_{\mathcal{D}}$, the Hermitian elements of \mathcal{A} represent the general observables of the particle.

A normalized element ψ of \mathcal{D} represents (up to a phase factor) a pure state of the particle. Given the particle in this state, the quantity

$$(41) \quad p(\Delta) \equiv \int_{\Delta} |\psi(x)|^2 dx$$

(where Δ is a Borel set in R^3) is interpreted as the probability that the particle lies in the domain Δ . For any observable $A \in \mathcal{A}$, the quantity

$$(42) \quad \langle A \rangle_\psi = (\psi, A\psi) \equiv \int \psi^*(x)(A\psi)(x) dx$$

represents the expectation value of A in the state ψ . With a suitable topology on the algebra \mathcal{A} [15], the quantity $\omega_\psi \equiv \langle . \rangle_\psi$ of Eq.(42) can be considered as a continuous linear functional on \mathcal{A} which is (i) positive (which means $\omega_\psi(B^*B) \geq 0 \quad \forall B \in \mathcal{A}$) and (ii) normalized (i. e. $\omega_\psi(I) = 1$). The set $\mathcal{S}(\mathcal{A})$ of continuous positive linear functionals on \mathcal{A} is closed under convex combinations [i.e. $\omega_i \in \mathcal{S}(\mathcal{A}) \Rightarrow \sum_i p_i \omega_i \in \mathcal{S}(\mathcal{A})$ with $p_i \geq 0, \sum_i p_i = 1$]. A nontrivial convex combination of pure states is called a mixed state or mixture.

It should now be easy to understand that a reasonably satisfactory way of presenting the traditional formalism of QM of a system (which permits

free use of unbounded observables) is to associate, with a quantum system S, a *quantum triple* $(\mathcal{H}, \mathcal{D}, \mathcal{A}_Q)$ where \mathcal{H} is a complex, separable Hilbert space (which may or may not be finite dimensional), \mathcal{D} a dense linear domain in \mathcal{H} (which is obviously equal to \mathcal{H} when \mathcal{H} is finite dimensional) and \mathcal{A}_Q an algebra of linear operators which, along with their adjoints, have \mathcal{D} as an invariant domain. For any $A \in \mathcal{A}_Q$, we define its conjugate as $A^* = A^\dagger|_{\mathcal{D}}$ (thus defining an involution $*$ on \mathcal{A}_Q). Observables of the system are the Hermitian elements of \mathcal{A}_Q .

For systems where a set of fundamental observables can be identified (like the one considered above), the algebra \mathcal{A}_Q is the one generated by the fundamental observables (and I) subject to appropriate commutation relations.

States of S are those density operators ρ such that

$$(43) \quad \langle A \rangle_\rho = \text{Tr}(\rho A)$$

is defined for all $A \in \mathcal{A}_Q$. For an observable A, the real quantity $\langle A \rangle_\rho$ represents the expectation value of A when S is in the state ρ . [Note. By states we strictly mean physical states so that expectation values of all observables are defined in all states.] Pure states are represented (up to phase factors of modulus one) by normalized vectors $\psi \in \mathcal{D}$ such that $\langle A \rangle_\psi = (\psi, A\psi)$. The density operator corresponding to a state ψ is $|\psi\rangle\langle\psi|$ in the Dirac notation.

Dirac bra and ket spaces can be introduced in terms of Gelfand triples [20] based on the pair $(\mathcal{H}, \mathcal{D})$; we shall, however, skip the details.

When the algebra \mathcal{A}_Q is ‘special’ (in the sense defined in section 1), one has a canonical form ω_c defined on it [see Eq.(35)]. The *quantum symplectic structure* is defined on \mathcal{A}_Q by employing the *quantum symplectic form*

$$(44) \quad \omega_Q = -i\hbar\omega_c.$$

Note that the factor i serves to make ω real and \hbar to give it the dimension of action (which is the correct dimension of a symplectic form in mechanics). The minus sign is a matter of convention. Eq.(38) now gives the *quantum Poisson bracket*

$$(45) \quad \{A, B\}_Q = (-i\hbar)^{-1}[A, B].$$

When the algebra \mathcal{A}_Q has both inner and outer derivations, one can employ the generalized symplectic algebra $(\mathcal{A}_Q, \text{IDer}(\mathcal{A}_Q), \omega_Q)$. Again, we have, for a given $A \in \mathcal{A}_Q$, $Y_A = (-i\hbar)^{-1}D_A$ and the quantum PB of Eq.(45).

A nontrivial center in \mathcal{A}_Q indicates the presence of superselection rules and/or external fields. We shall skip details on these matters.

3. The formalism of supmech

As mentioned earlier, supmech is an algebraic scheme of mechanics synthesizing noncommutative symplectic geometry and noncommutative probability. Most developments in it are parallel to those in classical Hamiltonian mechanics; in fact, it is essentially noncommutative Hamiltonian statistical mechanics with some extra sophistication in the treatment of states. In the detailed treatment in Ref. [11], the basic system algebra is taken to be a superalgebra (so as to provide a unified treatment of bosonic and fermionic objects/entities); here, however, we shall restrict ourselves to the simpler non-super version.

We shall call ‘experimentally accessible systems’ those on which repeatable experiments can be performed. For such systems, the statistical analysis of experiments can be done with the traditional frequency interpretation of probability. The universe as a whole and large subsystems of it on a cosmological scale obviously do not belong to this class. As of now, supmech has been developed only for experimentally accessible systems.

The essential points in the development of supmech are listed below.

1. The system algebra. Supmech associates with an experimentally accessible system S an ALGEBRA \mathcal{A} (its elements will be denoted as A, B, C, \dots). Hermitian elements of \mathcal{A} represent observables of S . We denote by $\mathcal{O}(\mathcal{A})$ the set of all observables in \mathcal{A} . (In fact, \mathcal{A} is assumed to be a locally convex algebra; we shall, however, not treat the topological aspects here.)

2. States. States of \mathcal{A} (denoted by the letters ϕ, ψ, \dots) are defined as (continuous) positive linear functionals which are normalized [i.e. $\phi(I) = 1$ where I is the unit element of \mathcal{A}]. The set $\mathcal{S}(\mathcal{A})$ of states of \mathcal{A} is clearly closed under convex combinations (weighted sums). Those states which cannot be represented as nontrivial convex combinations are called pure. The set of pure states of \mathcal{A} is denoted as $\mathcal{S}_1(\mathcal{A})$. For any $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$, the quantity $\phi(A)$ is to be interpreted as the expectation value of A in the state ϕ . When $A \in \mathcal{O}(\mathcal{A})$, $\phi(A)$ is, of course, real.

3. Compatible completeness of observables and pure states. The pair

$$(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$$

is assumed to be ‘compatibly complete’ in the sense that

- (i) given $A, B \in \mathcal{O}(\mathcal{A})$, $A \neq B$, there must be a pure state ϕ such that $\phi(A) \neq \phi(B)$;
- (ii) given two different pure states ϕ, ψ , there must be an observable A such that $\phi(A) \neq \psi(A)$.

We shall refer to this as the *CC condition*.

4. Symplectic structure on the system algebra. The system algebra is assumed to have a symplectic structure provided by a symplectic form ω . Symmetries of the formalism (the analogues of canonical transformations in classical Hamiltonian mechanics and unitary transformations in QM) are canonical transformations of the symplectic algebra (\mathcal{A}, ω) .

Note. The author has not opted for the economy that could be obtained by combining items (1) and (4) and introducing a system algebra directly as a symplectic algebra because the first three items above constitute a concrete unit serving a special purpose. [See remark (4) in the last section.]

5. Action of canonical transformations on states. Denoting the algebraic dual of the algebra \mathcal{A} by \mathcal{A}^* , an automorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ induces the dual/transpose mapping $\tilde{\Phi} : \mathcal{A}^* \rightarrow \mathcal{A}^*$ such that, in obvious notation,

$$(46) \quad \tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \quad \text{or} \quad \langle \tilde{\Phi}(\phi), A \rangle = \langle \phi, \Phi(A) \rangle$$

where \langle, \rangle denotes the dual space pairing. The mapping $\tilde{\Phi}$ maps states (which form a subset of \mathcal{A}^*) onto states. To see this, note that

- (i) $[\tilde{\Phi}(\phi)](A^*A) = \phi(\Phi(A^*A)) = \phi[\Phi(A)^*\Phi(A)] \geq 0$;
- (ii) $[\tilde{\Phi}(\phi)](I) = \phi[\Phi(I)] = \phi(I) = 1$.

The linearity of $\tilde{\Phi}$ (on \mathcal{A}^*) ensures that it preserves convex combinations of states. In particular, it maps pure states onto pure states. We have, therefore, a bijective mapping $\tilde{\Phi} : \mathcal{S}_1(\mathcal{A}) \rightarrow \mathcal{S}_1(\mathcal{A})$.

When Φ is a canonical transformation, we have, for $X, Y \in \text{Der}(\mathcal{A})$,

$$\omega(X, Y) = (\Phi^*\omega)(X, Y) = \Phi^{-1}[\omega(\Phi_*X, \Phi_*Y)]$$

giving

$$(47) \quad \Phi[\omega(X, Y)] = \omega(\Phi_*X, \Phi_*Y).$$

Taking expectation value of each side in the state ϕ , we get

$$(48) \quad \tilde{\Phi}(\phi)[\omega(X, Y)] = \phi[\omega(\Phi_* X, \Phi_* Y)].$$

Defining ω_Φ by

$$(49) \quad \omega_\Phi(X, Y) = \omega(\Phi_* X, \Phi_* Y)$$

we can write Eq.(48) as

$$(50) \quad (\tilde{\Phi}\phi)[\omega(., .)] = \phi[\omega_\Phi(., .)].$$

When Φ is an infinitesimal canonical transformation generated by $G \in \mathcal{A}$, we have

$$(51) \quad [\tilde{\Phi}(\phi)](A) \simeq \phi(A + \epsilon\{G, A\}).$$

Putting $\tilde{\Phi}(\phi) = \phi + \delta\phi$, we have

$$(52) \quad (\delta\phi)(A) = \epsilon\phi(\{G, A\}).$$

6. Dynamics. Dynamics of the system is described by a one-parameter family Φ_t of canonical transformations generated by an observable H called the *Hamiltonian*; the triple (\mathcal{A}, ω, H) will be called a *supmech Hamiltonian system*. As in QM or classical statistical mechanics, there are two standard ways of describing dynamics corresponding to the choice of making the evolution mappings act on observables (Heisenberg type picture) or states (Schrödinger type picture); the two pictures are related as [writing $\Phi_t(A) = A(t)$ and $\tilde{\Phi}_t(\phi) = \phi(t)$]

$$(53) \quad \langle \phi(t), A \rangle = \langle \phi, A(t) \rangle.$$

In the Heisenberg type picture we have

$$dA(t) = A(t + dt) - A(t) \simeq Y_H[A(t)]dt$$

giving the *Hamilton's equation* of supmech :

$$(54) \quad \frac{dA(t)}{dt} = Y_H[A(t)] = \{H, A(t)\}.$$

In the Schrödinger type picture, Eq.(52) with $\Phi = \Phi_t$ gives the *Liouville equation* of supmech :

$$(55) \quad \frac{d\phi(t)}{dt}(A) = \phi(t)(\{H, A\}) \quad \text{or} \quad \frac{d\phi(t)}{dt}(\cdot) = \phi(t)(\{H, \cdot\}).$$

7. Classical Hamiltonian mechanics and QM as subdisciplines of supmech.

(i) *Classical Hamiltonian mechanics.* Traditionally developed in the framework of a symplectic manifold (M, ω_{cl}) [41], it can be treated in supmech by taking $\mathcal{A} = C^\infty(M, \mathbb{C}) \equiv \mathcal{A}_{cl}$, the commutative algebra of smooth complex-valued functions on the phase space M. The observables of this systems are the subclass of real-valued functions. For the algebra \mathcal{A}_{cl} , the derivations are the smooth vector fields and the differential forms of section (1) are the traditional differential forms on the manifold M. The symplectic structure on \mathcal{A}_{cl} is given by the classical symplectic form on M given, in standard notation, by

$$\omega_{cl} = \sum_{j=1}^n dp_j \wedge dq^j$$

where $\dim(M) = 2n$. Writing, in terms of the general local coordinates ξ^a ($a=1, \dots, 2n$) on M, $\omega_{cl} = (\omega_{cl})_{ab} d\xi^a \wedge d\xi^b$, the supmech Poisson bracket on \mathcal{A}_{cl} is the classical Poisson bracket on M :

$$(56) \quad \{f, g\}_{cl} = \omega_{cl}^{ab} \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b} = \sum_j \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} \right)$$

where (ω_{cl}^{ab}) is the inverse of the matrix $((\omega_{cl})_{ab})$. The supmech Hamilton equation (54) is, in the present context, the traditional Hamilton's equation

$$(57) \quad \frac{df}{dt} = \{H_{cl}, f\}_{cl}.$$

States of \mathcal{A}_{cl} are probability measures on M; in obvious notation, they are of the form $\phi_\mu(f) = \int_M f d\mu$. Pure states are Dirac measures (or, equivalently, points of M) μ_{ξ_0} for which $\phi_{\xi_0}(f) = f(\xi_0)$.

The pair $(\mathcal{O}(\mathcal{A}_{cl}), \mathcal{S}_1(\mathcal{A}_{cl}))$ of classical observables and pure states is easily seen to be compatibly complete : Given two real-valued functions on M, there is a point of M at which they take different values and, given two different points of M, there is a real-valued function on M which takes different values at those points.

In ordinary mechanics, only pure states are used. More general states are used in classical statistical mechanics where, in most applications, they are taken to be represented by densities on M [$d\mu = \rho(\xi)d\xi$ where $d\xi = dqdp$ is the Liouville volume element on M]. The state evolution equation of supmech gives, in the present context,

$$\int_M \left(\frac{\partial \rho(\xi, t)}{\partial t} \right) (\xi) f(\xi) d\xi = \int_M \rho(\xi, t) \{H, f\}_{cl}(\xi) d\xi.$$

Taking $M = R^{2n}$, noting that the density ρ must vanish at infinity and performing a partial integration, the right hand side becomes $\int_M \{\rho, H\}_{cl} f d\xi$ giving the traditional Liouville equation

$$(58) \quad \frac{\partial \rho}{\partial t} = \{\rho, H\}_{cl}.$$

(ii) *Quantum mechanics.* Most of the needful has already been done in the previous section. Given a quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A}_Q)$, the supmech system algebra is to be taken as \mathcal{A}_Q . The family of pure states consists of unit rays corresponding to vectors in \mathcal{D} . The condition of compatible completeness of the pair $(\mathcal{A}_Q, \mathcal{S}_1(\mathcal{A}_Q))$ is easily verified :

(i) Given $A, B \in \mathcal{O}(\mathcal{A}_Q)$ and $(\psi, A\psi) = (\psi, B\psi)$ for all $\psi \in \mathcal{D}$, we have $(\phi, A\psi) = (\phi, B\psi)$ for all $\phi, \psi \in \mathcal{D}$ implying $A = B$. [Hint: Consider the given equality with state vectors $(\phi + \psi)/\sqrt{2}$ and $(\phi + i\psi)/\sqrt{2}$.]

(ii) Given normalized vectors $\phi, \psi \in \mathcal{D}$, and $(\phi, A\phi) = (\psi, A\psi)$ for all $A \in \mathcal{O}(\mathcal{A}_Q)$, the equality $\phi = \psi$ (up to a phase) can be seen by using the given equality with A taken as the projection operators corresponding to members of an orthonormal basis containing ψ as a member.

We have the quantum symplectic algebra $(\mathcal{A}_Q, \omega_Q)$ and the associated quantum Poisson brackets as in the previous section. The supmech Hamilton equation (54) in the present case is clearly the Heisenberg equation of motion

$$\frac{dA(t)}{dt} = \{H, A(t)\}_Q = (-i\hbar)^{-1}[H, A(t)].$$

The supmech Liouville equation (55) with the states given by density operators $\omega_\rho(A) = \text{Tr}(\rho A)$ gives the ‘quantum Liouville equation’ (or the von Neumann equation)

$$(59) \quad \frac{d\rho(t)}{dt} = (-i\hbar)^{-1}[\rho, H] = \{\rho(t), H\}_Q.$$

8. *Supmech as a framework for an autonomous development of QM*

In the traditional development of QM, one generally *quantizes* classical systems. For example, to obtain the Schrödinger equation

$$(60) \quad i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi \equiv H\psi$$

in the traditional treatment of the QM of a nonrelativistic spinless particle, one starts with the classical Hamiltonian

$$(61) \quad H = \frac{\mathbf{p}^2}{2m} + V,$$

introduces the Hilbert space $\mathcal{H} = L^2(R^3)$ of complex square integrable functions, prescribes rules for the replacement of the classical variables x_j and p_j by the operators X_j and P_j of Eq.(39) [thus obtaining the quantum Hamiltonian operator H of Eq.(60)] and finally (taking clue from the classical equation $H_{cl} = E$), prescribes the rule for the evolution equation for the Schrödinger wave function $\psi(\mathbf{x}, t)$ in the form $\hat{E}\psi = H\psi$ with $\hat{E} = i\hbar \frac{\partial}{\partial t}$.

In Ref. [9], the need for an autonomous development of QM was emphasized and some stringent criteria were laid down for such a development. In the framework of supmech, it is possible to develop the QM of particles autonomously satisfying those criteria [11]. We give here an outline of the steps involved in the autonomous development of the Schrödinger equation (60). [The idea is to define a particle as a localizable elementary system (which involves a discussion of the action of the appropriate relativity group on the system algebra and of localizable systems), have a systematic way to identify the fundamental observables of a particle and obtain an expression for the Hamiltonian (infinitesimal generator of time translations) in terms of the fundamental observables (which can be done group theoretically [2]), and have a systematic procedure to obtain a/the Hilbert space realization of the relevant dynamics.]

(i) One defines the *Poisson action* [41, [11] of a Lie group G on a symplectic algebra (\mathcal{A}, ω) as an assignment, to every element $g \in G$, a canonical transformation of the algebra such that the infinitesimal generators ('hamiltonians') of one-parameter subgroups of the canonical transformations have Poisson brackets in correspondence with the commutation relations in the Lie algebra of G .

(ii) The concept of a *localizable system* is introduced [as one which has a configuration space M (a topological space) associated with it and it is meaningful to talk about the probability of the system being localized in a Borel subset of M] in which the concept of a position/configuration observable naturally emerges. For systems with configuration space R^n , the concept of *concrete Euclidean-covariant localization* is introduced in which one has the position observables X_j and the Euclidean group generators P_j and $M_{jk}(= -M_{kj})$ satisfying the standard Poisson bracket relations.

(iii) For the subclass of supmech systems for which the concept of space and time and of a relativity scheme are relevant, the appropriate relativity scheme is implemented through the Poisson action of the corresponding relativity group G_0 on the system algebra. In the nonrelativistic case (Galilean relativity), the need for a Poisson action requires the replacement of the Galilean group G by its projective group [2] \hat{G} which is a central extension of the universal covering group of G . The additional generator corresponds to mass. In this manner, the concept of mass appears naturally for the system at the fundamental level.

For the implementation of a relativity scheme (with a relativity group G_0), it is useful to introduce the concept of the *effective relativity group* \hat{G}_0 which is the universal covering group \tilde{G}_0 of G_0 if the latter admits Poisson actions and the projective group \hat{G}_0 if it does not.

(iv) In supmech, an *elementary system* [for a given relativity scheme (or relativity group)] is defined (generalizing and extending the treatments of elementary systems by Wigner [40] and Alonso [2]) as a supmech triple $(\mathcal{A}, \omega, \mathcal{S}_1(\mathcal{A}))$ such that the effective relativity group \hat{G}_0 has a Poisson action on the symplectic algebra (\mathcal{A}, ω) and a transitive action on the space $\mathcal{S}_1(\mathcal{A})$ of pure states.

The fundamental observables of an elementary system are proposed to be identified from the PBs of the ‘hamiltonians’ coming from the effective relativity group \hat{G}_0 . For the Galilean elementary systems, they turn out to be M, X_j, P_j and S_j ($j=1,2,3$) corresponding, respectively, to mass, position, momentum and spin. For a spinless particle they are M, X_j and P_j . The observable M (mass) has zero PBs with all other observables. It is, therefore, a constant; its value m characterizes the elementary system and the objects X_j, P_j serve as kinematic observables. Simple group theory leads to the following general expression (for massive elementary systems) for the gen-

erator H (the Hamiltonian) of time translations in terms of the fundamental observables :

$$(62) \quad H = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}, \mathbf{P}).$$

(v) A Hilbert space realization of the supmech kinematics and dynamics of a system with noncommutative algebra, if it exists, is very much desirable because, in such a realization, the CC condition treated above is automatically satisfied (as was seen in the subsection **7** above); otherwise, one has to keep track of it separately. The existence of a Hilbert space realization is, in fact, guaranteed by the CC condition : there being a rich supply of (pure) states, one can employ the GNS construction (the version of it best suited for us is that of Ref.[24]) based on one of them to obtain a Hilbert space representation of the algebra \mathcal{A} . Such a representation is generally not faithful; for example, if the state chosen is one with zero expectation value for the kinetic energy (of a non-relativistic particle), the momentum operator in the resulting Hilbert space representation will be identically zero. The CC condition again comes to the rescue; a faithful representation can be obtained by taking an appropriate direct sum of the GNS representations of the above sort.

All this trouble is, however, not necessary — at least for a system consisting of a single particle. The condition of transitive action of \hat{G}_0 on pure states implies that a Hilbert space realization (in which pure states are vector states) must employ an irreducible representation of this group. This, combined with the points treated above, then ensures that the representation must be the Schrödinger representation. The probability interpretation of Schrödinger wave functions *follows* from the formalism. (This is because the essential relevant physics is covered by the treatment of localizability above. This is, in fact, very satisfying — the probability interpretation of ‘ ψ ’ is no longer mysterious.)

The supmech evolution equation for pure states, with the Hamiltonian of Eq.(62) (with V a function of X only in simple applications), gives the traditional Schrödinger equation. It should be noted that the classical Hamiltonian or Lagrangian for a particle was not used at any stage in this development.

Note. Apart from ensuring an autonomous development of QM and the interpretation of ‘ ψ ’ above, a couple of attractive features the formalism

outlined above are :

(1) The Planck constant \hbar has to be introduced ‘by hand’ only once — in the quantum symplectic form (the most natural place to do it); its appearance at other conventional places — the canonical commutation relations (40), the Heisenberg equation and the Schrödinger equation (60) — is then automatic.

(2) The Dirac bra-ket formalism (in its rigorous version) appears naturally in the present setting. It is this formalism — and not von Neumann’s formalism [30] employing bounded observables — which is used in most quantum mechanical work.

4. Interacting systems in supmech

In this section, we shall consider, in the framework of supmech, the interaction of two systems S_1 and S_2 described individually as the supmech Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ ($i=1,2$). We shall treat the coupled system $S_1 + S_2$ also as a supmech Hamiltonian system. To this end, we associate, with the coupled system $S_1 + S_2$ the (algebraic) tensor product algebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$. The most important job in the present section is, given the symplectic forms $\omega^{(1)}$ and $\omega^{(2)}$ on $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, to determine the symplectic form and the PB on \mathcal{A} .

4.1 The symplectic form and PB on the algebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$

The algebra $\mathcal{A}^{(1)}$ (resp. $\mathcal{A}^{(2)}$) has, in \mathcal{A} , an isomorphic copy consisting of the elements $(A \otimes I_2, A \in \mathcal{A}^{(1)})$ (resp. $I_1 \otimes B, B \in \mathcal{A}^{(2)}$) to be denoted as $\tilde{\mathcal{A}}^{(1)}$ (resp. $\tilde{\mathcal{A}}^{(2)}$) where I_1 and I_2 are the unit elements of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively. We shall also use the notations $\tilde{A}^{(1)} \equiv A \otimes I_2$ and $\tilde{B}^{(2)} \equiv I_1 \otimes B$.

Objects in $\mathcal{A}^{(i)}$ and $\tilde{\mathcal{A}}^{(i)}$ are related through the induced mappings corresponding to the isomorphisms $\Xi^{(i)} : \mathcal{A}^{(i)} \rightarrow \tilde{\mathcal{A}}^{(i)}$ ($i=1,2$) given by $\Xi^{(1)}(A) = A \otimes I_2$ and $\Xi^{(2)}(B) = I_1 \otimes B$. In particular

(i) The induced mapping $\Xi_*^{(1)} : Der(\mathcal{A}^{(1)}) \rightarrow Der(\tilde{\mathcal{A}}^{(1)})$ gives $\Xi_*^{(1)}(X) = \tilde{X}^{(1)}$ where

$$\tilde{X}^{(1)}(\tilde{A}^{(1)}) = \Xi^{(1)}[X(A)] = X(A) \otimes I_2.$$

Similarly, corresponding to $Y \in Der(\mathcal{A}^{(2)})$, we have $\tilde{Y}^{(2)} \in Der(\tilde{\mathcal{A}}^{(2)})$ given by $\tilde{Y}^{(2)}(\tilde{B}^{(2)}) = I_1 \otimes Y(B)$.

(ii) The induced mappings on 1-forms give, corresponding to the 1-forms $\alpha \in \Omega^1(\mathcal{A}^{(1)})$ and $\beta \in \Omega^1(\mathcal{A}^{(2)})$, we have $\tilde{\alpha}^{(1)} \in \Omega^1(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{\beta}^{(2)} \in \Omega^1(\tilde{\mathcal{A}}^{(2)})$ given by

$$\tilde{\alpha}^{(1)}(\tilde{X}^{(1)}) = \Xi^{(1)}[\alpha(([\Xi^{(1)}]^{-1})_* \tilde{X}^{(1)})] = \Xi^{(1)}[\alpha(X)] = \alpha(X) \otimes I_2$$

and $\tilde{\beta}^{(2)}(\tilde{Y}^{(2)}) = I_1 \otimes \beta(Y)$. Similar formulas hold for the higher forms.

To obtain the general differential forms and the exterior derivative on \mathcal{A} , the most straightforward procedure is to obtain the graded differential space $(\Omega(\mathcal{A}), d)$ as the tensor product [21] of the graded differential spaces $(\Omega(\mathcal{A}^{(1)}), d_1)$ and $(\Omega(\mathcal{A}^{(2)}), d_2)$. A differential k-form on \mathcal{A} is of the form (in obvious notation)

$$\alpha_k = \sum_{i+j=k} \alpha_i^{(1)} \otimes \alpha_j^{(2)}.$$

The exterior derivative d on $\Omega(\mathcal{A})$ is given by [here $\alpha \in \Omega^p(\mathcal{A}^{(1)})$ and $\beta \in \Omega(\mathcal{A}^{(2)})$]

$$(63) \quad d(\alpha \otimes \beta) = (d_1 \alpha) \otimes \beta + (-1)^p \alpha \otimes (d_2 \beta).$$

Given the symplectic forms $\omega^{(i)}$ on $\mathcal{A}^{(i)}$ ($i=1,2$) and stipulating that the symplectic form ω on \mathcal{A} should not depend on anything other than the objects $\omega^{(i)}$ and $I_{(i)}$ ($i=1,2$) (the ‘naturalness’/‘canonicity’ assumption), the only possible choice of ω is

$$(64) \quad \omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}.$$

To show that it is, indeed, a symplectic form, we must show that it is (i) closed and (ii) non-degenerate. Eq.(63) gives

$$d\omega = (d_1 \omega^{(1)}) \otimes I_2 + \omega^{(1)} \otimes d_2(I_2) + d_1(I_1) \otimes \omega^{(2)} + I_1 \otimes d_2 \omega^{(2)} = 0$$

showing that ω is closed.

To show the non-degeneracy of ω , we must show that, given $A \otimes B \in \mathcal{A}$, there exists a unique derivation $Y = Y_{A \otimes B}$ in $Der(\mathcal{A})$ such that

$$(65) \quad \begin{aligned} i_Y \omega = -d(A \otimes B) &= -(d_1 A) \otimes B - A \otimes (d_2 B) \\ &= i_{Y_A^{(1)}} \omega^{(1)} \otimes B + A \otimes i_{Y_B^{(2)}} \omega^{(2)}. \end{aligned}$$

The structure of Eq.(65) suggests that Y must be of the form

$$(66) \quad Y = Y_A^{(1)} \otimes \Psi_B^{(2)} + \Psi_A^{(1)} \otimes Y_B^{(2)}$$

where $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ are linear mappings on $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively such that $\Psi_A^{(1)}(I_1) = A$ and $\Psi_B^{(2)}(I_2) = B$. A general object of the form (66), however, need not be a derivation of \mathcal{A} ; we must, therefore, impose the condition that Y must be a derivation. Recalling Eq.(1) and denoting the multiplication operators in $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ and \mathcal{A} by μ_1, μ_2 and μ respectively, we have

$$(67) \quad Y \circ \mu(C \otimes D) - \mu(C \otimes D) \circ Y = \mu(Y(C \otimes D)).$$

Noting that $\mu(C \otimes D) = \mu_1(C) \otimes \mu_2(D)$, Eq.(67) with Y of Eq.(66) gives

$$(68) \quad \begin{aligned} & (Y_A^{(1)} \circ \mu_1(C)) \otimes (\Psi_B^{(2)} \circ \mu_2(D)) + (\Psi_A^{(1)} \circ \mu_1(C)) \otimes (Y_B^{(2)} \circ \mu_2(D)) \\ & - (\mu_1(C) \circ Y_A^{(1)}) \otimes (\mu_2(D) \circ \Psi_B^{(2)}) - (\mu_1(C) \circ \Psi_A^{(1)}) \otimes (\mu_2(D) \circ Y_B^{(2)}) \\ & = \mu[Y_A^{(1)}(C) \otimes \Psi_B^{(2)}(D) + \Psi_A^{(1)}(C) \otimes Y_B^{(2)}(D)]. \end{aligned}$$

Since $Y_A^{(1)}$ and $Y_B^{(2)}$ are derivations, we must have

$$(69) \quad \begin{aligned} & Y_A^{(1)} \circ \mu_1(C) - \mu_1(C) \circ Y_A^{(1)} = \mu_1(Y_A^{(1)}(C)) = \mu_1(\{A, C\}_1) \\ & Y_B^{(2)} \circ \mu_2(D) - \mu_2(D) \circ Y_B^{(2)} = \mu_2(Y_B^{(2)}(D)) = \mu_2(\{B, D\}_2). \end{aligned}$$

Putting $D = I_2$ in Eq.(68), we have [noting that $\mu_2(D) = \mu_2(I_2) = id_2$, the identity mapping on $\mathcal{A}^{(2)}$ and $Y_B^{(2)}(I_2) = 0$]

$$\begin{aligned} & (Y_A^{(1)} \circ \mu_1(C)) \otimes \Psi_B^{(2)} + (\Psi_A^{(1)} \circ \mu_1(C)) \otimes Y_B^{(2)} \\ & - (\mu_1(C) \circ Y_A^{(1)}) \otimes \Psi_B^{(2)} - (\mu_1(C) \circ \Psi_A^{(1)}) \otimes Y_B^{(2)} \\ & = \mu[Y_A^{(1)}(C) \otimes B] = \mu_1(\{A, C\}_1) \otimes \mu_2(B) \end{aligned}$$

which, along with Eq.(69), gives

$$(70) \mu_1(\{A, C\}_1) \otimes [\Psi_B^{(2)} - \mu_2(B)] = [\mu_1(C) \circ \Psi_A^{(1)} - \Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)}.$$

Similarly, putting $C = I_1$ in Eq.(68), we get

$$(71) [\Psi_A^{(1)} - \mu_1(A)] \otimes \mu_2(\{B, D\}) = Y_A^{(1)} \otimes [\mu_2(D) \circ \Psi_B^{(2)} - \Psi_B^{(2)} \circ \mu_2(D)].$$

Now, equations (71) and (70) give

$$(72) \quad \Psi_A^{(1)} - \mu_1(A) = \lambda_1 Y_A^{(1)}$$

$$(73) \quad \mu_2(D) \circ \Psi_B^{(2)} - \Psi_B^{(2)} \circ \mu_2(D) = \lambda_1 \mu_2(\{B, D\}_2)$$

$$(74) \quad \Psi_B^{(2)} - \mu_2(B) = \lambda_2 Y_B^{(2)}$$

$$(75) \quad \mu_1(C) \circ \Psi_A^{(1)} - \Psi_A^{(1)} \circ \mu_1(C) = \lambda_2 \mu_1(\{A, C\}_1)$$

where λ_1 and λ_2 are complex numbers.

Equations (66), (72) and (74) give

$$(76) \quad \begin{aligned} Y &= Y_A^{(1)} \otimes [\mu_2(B) + \lambda_2 Y_B^{(2)}] + [\mu_1(A) + \lambda_1 Y_A^{(1)}] \otimes Y_B^{(2)} \\ &= Y_A^{(1)} \otimes \mu_2(B) + \mu_1(A) \otimes Y_B^{(2)} + (\lambda_1 + \lambda_2) Y_A^{(1)} \otimes Y_B^{(2)}. \end{aligned}$$

Note that only the combination $(\lambda_1 + \lambda_2) \equiv \lambda$ appears in Eq.(76). To have a unique Y , we must obtain an equation fixing λ in terms of given quantities.

Substituting for $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ from equations (72) and (74) into equations (73) and (75) and using equations (69), we obtain the equations

$$(77) \quad \lambda \mu_1(\{A, C\}_1) = \mu_1([C, A]) \quad \text{for all } A, C \in \mathcal{A}^{(1)}$$

$$(78) \quad \lambda \mu_2(\{B, D\}_2) = \mu_2([D, B]) \quad \text{for all } B, D \in \mathcal{A}^{(2)}.$$

We have not one but two equations of the type we have been looking for. This is a signal for the emergence of nontrivial conditions (for the desired symplectic structure on the tensor product algebra to exist).

Let us consider the equations (77,78) for the various possible situations:

(i) Let the algebra $\mathcal{A}^{(1)}$ be commutative. Assuming the PB $\{, \}_1$ is nontrivial, Eq.(77) implies that $\lambda = 0$. Then Eq.(78) implies that the algebra $\mathcal{A}^{(2)}$ is also commutative. It follows that

(a) when both the algebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are commutative, the unique Y is given by Eq.(76) with $\lambda = 0$;

(b) a ‘natural’/‘canonical’ symplectic structure does not exist on the tensor product of a commutative and a noncommutative algebra.

(ii) Let the algebra $\mathcal{A}^{(1)}$ be noncommutative. Eq.(77) then implies $\lambda \neq 0$ which, in turn, implies, through Eq.(78), that the algebra $\mathcal{A}^{(2)}$ is also noncommutative [which is also expected from (b) above]. Equations (77, 78) now give

$$(79) \quad \{A, C\}_1 = -\lambda^{-1}[A, C], \quad \{B, D\}_2 = -\lambda^{-1}[B, D]$$

which shows that when both the algebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are noncommutative, a ‘natural’/‘canonical’ symplectic structure on their tensor product exists if and only if each algebra has a quantum symplectic structure with the *same* parameter $(-\lambda)$, i.e.

$$(80) \quad \omega^{(1)} = -\lambda\omega_c^{(1)}, \quad \omega^{(2)} = -\lambda\omega_c^{(2)}$$

where $\omega_c^{(1)}$ and $\omega_c^{(2)}$ are the canonical symplectic forms on the two algebras. It follows that *all noncommutative system algebras must have a universal quantum symplectic structure*. Comparison of Eq.(80) with the quantum symplectic form (44) shows that $\lambda = i\hbar$.

In all the permitted cases, the PB on the algebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ is given by

$$(81) \quad \begin{aligned} \{A \otimes B, C \otimes D\} &= \{A, C\}_1 \otimes BD + AC \otimes \{B, D\}_2 \\ &+ \lambda \{A, C\}_1 \otimes \{B, D\}_2 \end{aligned}$$

where the parameter λ vanishes in the commutative case; in the noncommutative case, it is the universal parameter appearing in the symplectic forms (80).

In Ref. [10], the following PB was reported for the tensor product algebra \mathcal{A} :

$$(82) \quad \{A \otimes B, C \otimes D\} = \{A, B\}_1 \otimes \frac{CD + DC}{2} + \frac{AC + CA}{2} \otimes \{C, D\}_2.$$

When both the algebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are commutative, the equations (81) (with $\lambda = 0$) and (82) are clearly the same. In fact, the same is also true when both the algebras are noncommutative. To see this, it is adequate to note that, using Eq.(79), we have

$$\begin{aligned} \lambda \{A, C\}_1 \otimes \{B, D\}_2 &= [C, A] \otimes \{B, D\}_2 = \{A, C\}_1 \otimes [D, B] \\ &= \frac{CA - AC}{2} \otimes \{B, D\}_2 + \{A, C\}_1 \otimes \frac{DB - BD}{2}. \end{aligned}$$

In Ref. [10], the PB of Eq.(82) was meant to be true for the general case which includes the case when one of the two algebras is commutative and the other noncommutative (the mixed case). Shortly after the paper in Ref.[10] appeared in the arXiv, M.J.W. Hall, in a private communication to the author, pointed out that the ‘Poisson bracket’ (82) does not satisfy the

Jacobi identity in some cases (as shown, for example, in Ref.[6]). The present work is an outcome of the efforts to clarify the situation in this matter.

The example of violation of the Jacobi identity belonged to the mixed case. We now know that, in this case, Eq.(64) does not represent a valid symplectic structure. The mistake in the earlier work of the author consisted in not ensuring that the Y of Eq.(66) is a derivation.

Comment on a possible generalized symplectic structure (of the type mentioned in section 1.5) is being postponed to the last section (item 5 there).

4.2 Dynamics of the interacting system. Given that the two systems S_1 and S_2 are represented as supmech Hamiltonian systems as mentioned above, the coupled system $S_1 + S_2$ is also to be represented as a supmech Hamiltonian system (\mathcal{A}, ω, H) with the symplectic form ω as in Eq.(64) and the Hamiltonian given by

$$(83) \quad H = H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + H_{int}.$$

In most applications, the interaction hamiltonian is of the form

$$(84) \quad H_{int} = \sum_{i=1}^n F_i \otimes G_i$$

where F_i and G_i are observables of the two systems and the coupling constants have been absorbed in these observables.

In the Heisenberg type picture, evolution of a typical observable $A(t) \otimes B(t)$ is governed by the supmech Hamilton equation

$$(85) \quad \begin{aligned} \frac{d}{dt}[A(t) \otimes B(t)] &= \{H, A(t) \otimes B(t)\} \\ &= \{H^{(1)}, A(t)\}_1 \otimes B(t) + A(t) \otimes \{H^{(2)}, B(t)\}_2 \\ &\quad + \{H_{int}, A(t) \otimes B(t)\}. \end{aligned}$$

In Ref.[10], this formalism was applied to the treatment of measurements in quantum mechanics taking S_1 to be the measured quantum system and S_2 the apparatus (assumed macroscopic) treated as a classical system and using the PB of Eq.(82). Since a quantum-classical interaction is now not permitted, we must go back to the original von Neumann idea [30] to treat the apparatus as a quantum mechanical system. We are, however, not constrained to adopt the von Neumann procedure [30], [38] of introducing vector

states for the pointer positions (which is the basic cause of all the problems in quantum measurement theory). Here supmech offers an advantage not available in von Neumann's treatment. Since both quantum and classical systems can be accommodated in the supmech formalism, one can exploit the fact that the apparatus can be described classically to a very good approximation. The best way to do this is to use the phase space description of the QM of the apparatus (the Weyl-Wigner-Moyal formalism [37], [39], [29]) and then go to the classical approximation (doing it all within the supmech formalism). With such a modification, the program of Ref[10] goes through successfully, justifying the final results obtained there. We shall, however, skip the details here.

5. Concluding remarks

1. Supmech permits two kinds of 'worlds' : the *commutative world* in which all system algebras are commutative and the *noncommutative world* in which they are all noncommutative. There is no restriction (as far as the consistency of the supmech formalism is concerned) on the possible symplectic structures on system algebras in the commutative world; however, the system algebras in the noncommutative world must all have a universal quantum symplectic structure. Since QM is known to describe systems in nature substantially correctly, the real world is, of course, noncommutative; systems in the commutative world can appear only as approximations to those in the real quantum world.

2. The existence of a natural place for a universal Planck-like constant in the formalism is an important feature of supmech and deserves further comment and elaboration.

In physics, out of the three fundamental constants G (Newton's constant of gravitation), c (the speed of light in vacuum) and \hbar (the Planck constant), the first (G) appears in the statement of a universal law of nature (in Newton's law of gravitation and in Einstein's gravitational field equation in the general theory of relativity); the second (c) appears in classical electromagnetic theory as the speed of electromagnetic waves in vacuum, it is *postulated* as a universal speed in special relativity and maintains such existence in general relativity through the equivalence principle. The last one (\hbar) was introduced in the relation $E = h\nu = \hbar\omega$ as the proportionality constant

between energy and frequency in the hypothesized fundamental unit (‘quantum’) of energy in the energy exchange between interacting systems; in the traditional development of QM, it is put ‘by hand’ in various equations — the canonical commutation relations, the Heisenberg’s equation of motion and the Schrödinger equation. As has been already mentioned, QM is need of a proper formalism. The fact that supmech, apart from its geometrical setting and other appealing features, predicts the existence of a universal Planck-like constant, is a strong indication that the ‘right’ formalism has been chosen for an autonomous development of QM.

3. If one could construct a formalism in which there are *similar* natural places for three independent dimensional parameters [say, \hbar , c and l (a fundamental length)], it would constitute substantial progress towards construction of the ‘theory of everything’. For this, one might try to find sub-theories of supmech with natural places for c and l or, more generally, supmech-like theories which have the above-mentioned feature of supmech for all the three parameters.

Emphasis on the word ‘similar’ in the previous para means that the other two universal constants should also appear as proportionality constants in the choices of appropriate geometrical objects as multiples of the corresponding ‘canonical’ objects (recall $\omega_Q = -i\hbar\omega_c$) — or through some similar compelling geometrical reasoning. If we relax this requirement, one can find other ways of having reasonably ‘natural’ looking places for universal constants which may not have as profound implications as the appearance of the parameter \hbar had in supmech. For example, one may choose to work in a spatial lattice of fundamental spacing ‘ a ’ and employ discrete evolution with step length ‘ b ’ of the evolution parameter (‘time’); one can now take $l = a$ and $c = a/b$. While such a scheme may be of value, this is not what the author meant in the previous para.

4. The first three items in section 3 (relating to observables, states and the CC condition) were planned to constitute a reasonably standardized noncommutative probabilistic setting which, as we have seen, holds promise for being the proper replacement of the deterministic setting of classical mechanics for the description of dynamics of systems and, more generally, for probability theoretic developments.

5. In the ‘mixed’ case, when one of the algebras, say $\mathcal{A}^{(1)}$, is commutative and the other noncommutative, it is possible to have a generalized symplectic structure (of the type mentioned in section 1.5). Writing fA for $f \otimes A$, a

general element of the tensor product algebra \mathcal{A} is of the form $\sum f_i A_i$ (finite sum); the product in \mathcal{A} takes the form

$$(\sum_i f_i A_i)(\sum_j g_j B_j) = \sum_{i,j} f_i g_j A_i B_j.$$

The subalgebra $\tilde{\mathcal{A}}^{(1)}$ belongs to the center of \mathcal{A} . Taking, in the notation of section 1.5, $\mathcal{X} = IDer(\mathcal{A})$, we can have the generalized symplectic algebra $(\mathcal{A}, \mathcal{X}, \omega)$ with $\omega = b\omega_c$ giving the PB

$$(86) \quad \{fA, gB\} = b^{-1}fg[A, B].$$

When $\mathcal{A}^{(1)}$ represents a classical system and $\mathcal{A}^{(2)}$ a quantum one, such an approach clearly amounts to treating the classical observables as external fields. This is not adequate for a proper treatment of the interaction of a classical and a quantum system.

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